



Discrete Mathematics 2025 Spring



魏可佶 kejiwei@tongji.edu.cn



- 3.1 Basic Concepts of First-Order Logic
- 3.2 Equivalence Calculus of First-Order Logic

- 3.1.1 The Limitations of Propositional Logic
- 3.1.2 Individual Terms, Predicates, and Quantifiers
- 3.1.3 Symbolization of First-Order Logic
- 3.1.4 First-Order Logic Formulas and Classification

↳ The key limitations of propositional logic

- No representation of individuals
 - - Treats statements as whole units
 - - No variables (x) or predicates ($H(x)$, $M(x)$)
 - Example: Cannot express $\forall x (H(x) \rightarrow M(x))$

- No generalization or instantiation
 - - No quantifiers (\forall , \exists)
 - - Cannot infer from general to specific
 - Example: From $\forall x (H(x) \rightarrow M(x))$ and $H(s)$, cannot infer $M(s)$

↳ The key limitations of propositional logic (cont.)

■ Lack of expressive power in predicate logic:

Propositional logic cannot describe relationships between different individuals (e.g., parent-child relationships, size comparisons).

- For example, it cannot express the existential statement "Some people are mortal" $\exists xM(x)$.

e.g. >>> Example: Theorem proof

Suppose we have a mathematical theorem:

If a number is even, then it is divisible by 2.

Specific proposition: "4 is even."

Conclusion: "4 is divisible by 2."

↳ The key limitations of propositional logic (cont.)

- In first-order logic:

$\forall x(E(x) \rightarrow D(x))$ (All even numbers are divisible by 2).

$E(4)$ (4 is even).

Conclusion: $D(4)$ (4 is divisible by 2).

- Using propositional logic:

Let P represent "All even numbers are divisible by 2."

Let Q represent "4 is even."

Let R represent "4 is divisible by 2."

The reasoning takes the form: $(P \wedge Q) \rightarrow R$.

- Problem:

P is just a single proposition and cannot analyze its internal quantification $\forall x(E(x) \rightarrow D(x))$

It cannot perform instantiation reasoning (e.g., it cannot directly derive "4 is divisible by 2" from "All even numbers are divisible by 2").

- 3.1.1 The Limitations of Propositional Logic
- 3.1.2 Individual Terms, Predicates, and Quantifiers
 - Individual Constants, Individual Variables, Domain of Discourse, Universe of Discourse
 - Propositional Constants, Propositional Variables
 - Universal Quantifiers, Existential Quantifiers
- 3.1.3 Symbolization of First-Order Logic
- 3.1.4 First-Order Logic Formulas and Classification

↳ Definition of an individual term

- **Individual term:** Objects, either concrete or abstract, that can exist independently within the scope of study.
- **Individual constant:** An individual term representing specific things, denoted by a , b , c , etc.
- **Individual variable:** An individual term representing abstract things, denoted by x , y , z , etc.
- **Domain of individuals:** The range of values that individual variables can take.
- **Universal domain:** All things in the universe.
- **Such as:** "If x is even, then x is divisible by 2."
 - x , even, and 2 are individual terms; even and 2 are individual constants; x is an individual variable.
 - The domain of individuals can be the set of natural numbers N , the set of integers Z , etc., or it can be the universal domain.

↳ Definition of a Predicate

- **Predicate:** A word that represents the properties of individual terms or the relationships among them. Predicates are typically denoted by symbols such as F , G , H , and P .
- **Predicate constant:** A predicate that represents a **specific** property or relationship.
- **Predicate variable:** A predicate that represents an **abstract** property or relationship.
- **n -ary predicate $P(x_1, x_2, \dots, x_n)$:** A predicate with n propositional variables, defined as an n -ary function over the domain of individuals with a range of $\{0, 1\}$.
- **Unary predicate:** Represents properties of things.
- **Multi-ary predicate ($n \geq 2$):** Represents relationships among things.
- **0-ary predicate:** A predicate without individual variables, i.e., a propositional constant or propositional variable.

↳ Predicate example

(1) 4 is even

① "4" is an individual constant. ② "is even" is a predicate constant. ③ "4 is even" is symbolized as $F(4)$.

(2) Xiao Wang and Xiao Li are the same age

① "Xiao Wang" and "Xiao Li" are individual constants. ② "are the same age" is a predicate constant. ③ Let a : Xiao Wang, b : Xiao Li, and $G(x,y)$: x and y are the same age. ④ Symbolization as $G(a,b)$.

(3) $x < y$

① x and y are propositional variables. "<" is a predicate constant. ② Symbolization as $L(x,y)$.

(4) x has a certain property P

① x is a propositional variable. ② P is a predicate variable. ③ Symbolization as $P(x)$.

↳ Predicate example (cont.)

e.g. >>> **Example:** Symbolize the following propositions using 0th-order predicates and discuss their truth values.

(1) $\sqrt{2}$ is irrational, and $\sqrt{3}$ is rational

(2) If $2 > 3$, then $3 < 4$

■ **Solve:**

(1) Let $F(x)$: x is irrational, $G(x)$: x is rational

Symbolization as: $F(\sqrt{2}) \wedge G(\sqrt{3})$, True value: 0

(2) Let $F(x,y)$: $x > y$, $G(x,y)$: $x < y$,

Symbolization as: $F(2, 3) \rightarrow G(3, 4)$, True value: 1

↳ Quantifiers

- **Quantifiers:** Words that indicate quantity.
- **Universal quantifier \forall :** Indicates 'any', 'all', 'every', etc.
Such as: $\forall x$ means "for all x in the domain of discourse."
 $\forall xF(x)$ means "All x have the property F ."
- **Existential quantifier \exists :** Indicates 'there exists', 'there is', 'at least one', etc.

Such as:

$\exists x$ means "there exists an x in the domain of discourse."


$\exists xF(x)$ means "There exists an x that has the property F ."

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↳ Property Predicate vs. Unary Predicate

- A *property predicate* can be regarded as a subset or a special case of a *unary predicate*, i.e., $Property\ Predicate \subseteq Unary\ Predicate$.

Item	Property Predicate $P(x)$	Unary Predicate $P(x)$
Definition	Describes an individual's inherent property or characteristic	Any predicate that applies to a single variable
Relation	Describes only the property of an object , without involving other objects	May describe an object's property or its role/relationship
Examples	Even(x) (x is even) Red(x) (x is red)	Student(x) (x is a student) Parent(x) (x is a parent)
Application	used to express individual properties	A general term for any predicate with one parameter

 In machine learning model design, **property predicates** are typically used for classification tasks and can be utilized in **feature selection**, while **unary predicates** may involve object roles and **relational inference**.

↳ Property Predicate vs. First-order Logic(e.g.)

e.g. >>> **Examples:** Symbolizing the following propositions in first-order logic:

(1) All people love beauty; (2) There is someone who writes with their left hand. **Domain of discourse:** (a) The set of humans; (b) The universal domain.

■ **Solution: Domain of (a) :**

(1) Let $F(x)$: x loves beauty. Symbolized as $\forall x F(x)$.

(2) Let $G(x)$: x writes with their left hand. Symbolized as $\exists x G(x)$.

■ **Solution: Domain of (b) :** Let $M(x)$: x is a person. $F(x)$ and $G(x)$ are the same as in (a).

(1) Symbolized as $\forall x (M(x) \rightarrow F(x))$.

(2) Symbolized as $\exists x (M(x) \wedge G(x))$.

■ $M(x)$ is called a characteristic predicate.

↳ Property Predicate (e.g.)

e.g. >>> **Examples:** Symbolize the following propositions and discuss their truth values:

(1) For all x , $x^2 - 3x + 2 = (x - 1)(x - 2)$; (2) There exists an x such that $x + 5 = 3$.

Consider the following domains of discourse: (a) $D_1 = \mathbb{N}$ (the set of natural numbers); (b) $D_2 = \mathbb{R}$ (the set of real numbers).

■ **Solution:** Let $F(x): x^2 - 3x + 2 = (x - 1)(x - 2)$, and $G(x): x + 5 = 3$.

• (a) Domain $D_1 = \mathbb{N}$:

(1) $\forall x F(x)$ Truth value: 1 (True)

(2) $\exists x G(x)$ Truth value: 0 (False)

• (b) Domain $D_2 = \mathbb{R}$:

(1) $\forall x F(x)$ Truth value: 1 (True)

(2) $\exists x G(x)$ Truth value: 1 (True)